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ARCHIMEDEAN-LIKE CLASSES OF LATTICE-ORDERED GROUPS(1)

BY

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ABSTRACT. Suppose $\mathcal{C}$ denotes a class of totally ordered groups closed under taking subgroups and quotients by o-homomorphisms. We study the following classes: (1) Res $(\mathcal{C})$, the class of all lattice-ordered groups which are subdirect products of groups in $\mathcal{C}$; (2) Hyp $(\mathcal{C})$, the class of lattice-ordered groups in Res $(\mathcal{C})$ having all their l-homomorphic images in Res $(\mathcal{C})$; Para $(\mathcal{C})$, the class of lattice-ordered groups having all their principal convex l-subgroups in Res $(\mathcal{C})$. If $\mathcal{C}$ is the class of archimedean totally ordered groups then Para $(\mathcal{C})$ is the class of archimedean lattice-ordered groups, Res $(\mathcal{C})$ is the class of subdirect products of reals, and Hyp $(\mathcal{C})$ consists of all the hyper archimedean lattice-ordered groups.

We show that under an extra (mild) hypothesis, any given representable lattice-ordered group has a unique largest convex l-subgroup in Hyp $(\mathcal{C})$; this so-called hyper-$\mathcal{C}$-kernel is a characteristic subgroup. We consider several examples, and investigate properties of the hyper-$\mathcal{C}$-kernels.

For any class $\mathcal{C}$ as above we show that the free lattice-ordered group on a set X in the variety generated by $\mathcal{C}$ is always in Res $(\mathcal{C})$. We also prove that Res $(\mathcal{C})$ has free products.

Introduction. The theory presented in this paper grew out of an attempt to abstract the properties of the class of archimedean lattice-ordered groups (henceforth: l-groups), and some of its subclasses. The classes we construct offer an appetizing alternative to studying varieties of l-groups, one which quite surely has generalizations in other universal algebraic settings, although the reader may satisfy himself as to the special nature of many of the arguments.

For any class $\mathcal{C}$ of totally ordered groups (henceforth: o-groups) which is closed under taking subgroups and quotients by o-homomorphisms, we construct Res $(\mathcal{C})$, the class of l-groups which are subdirectly representable in a product of o-groups in $\mathcal{C}$. An l-group is hyper-$\mathcal{C}$ if it is in Res $(\mathcal{C})$ and every l-homomorphic image is in Res $(\mathcal{C})$. The main result in §1 says that every representable l-group contains a unique maximal convex l-subgroup which is hyper-$\mathcal{C}$; we call this the

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(1) This paper is dedicated to Donald A. Chambless; the idea of hyper-kernels is his, and in any case the author is indebted to him for his many suggestions and his patience through long hours of discussion on this material.

hyper-$\mathcal{C}$-kernel; it is a characteristic subgroup (Theorem 1.6).

In §2 we define Para($\mathcal{C}$): an l-group is in Para($\mathcal{C}$) if every principal convex l-subgroup is in Res($\mathcal{C}$). We show that if $\mathcal{C}$ is the class of archimedean o-groups then Para($\mathcal{C}$) is precisely the class of all archimedean l-groups (Proposition 2.1). Para($\mathcal{C}$) carries, in general, a lot of the abstract properties of archimedean l-groups: it is always closed under l-subgroups, products and quotients by polar subgroups (Proposition 2.2 and Theorem 2.4).

Our most surprising result is perhaps Theorem 3.5: for any set X, the free l-group over X in the variety generated by a class $\mathcal{C}$ is residually-$\mathcal{C}$. We also show that Res($\mathcal{C}$) has free products (Proposition 3.6). The crucial tool in §3 is that of the $\mathcal{C}$-radical of an l-group; it turns out to be the intersection of prime subgroups whose quotients in the l-group are $\mathcal{C}$-groups.

For the basic material on l-groups we refer the reader to [4] and [6]. We assume that the reader is familiar with the concept of a variety of universal algebras, or equivalently an equationally closed class of algebras. In any case our main reference for facts concerning universal algebra is [2].

Finally, we mention some special notation. For an l-group G, and $0 < g \in G$, $G(g)$ denotes the convex l-subgroup generated by g. If $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family of l-groups their *cardinal sum* G is their direct sum qua groups with coordinatewise ordering; the notation is $G = \boxplus \{G_\lambda \mid \lambda \in \Lambda\}$. If A is a (proper) subset of B we write $(A \subset B)$ $A \subseteq B$. If both A and B are subsets of a set X, $A \backslash B$ is the complement of B in A. For a subset S of a group G, $\langle S \rangle$ is the subgroup generated by S. If $g, h \in G$, $g^h = -h + g + h$.

1. **Hyper-$\mathcal{C}$-kernels.** The starting point in this discussion shall be a class of o-groups $\mathcal{C}$ which is closed under taking subgroups and quotients by o-homomorphisms. That is, if $G \in \mathcal{C}$ and H is o-isomorphic to a subgroup of G then $H \in \mathcal{C}$; further, if K is a normal, convex subgroup of G then $G/K \in \mathcal{C}$. For the duration of the article we shall deal only with classes of o-groups having these two closure properties. The model for our discussion is in the class of archimedean o-groups.

For a given class $\mathcal{C}$, an l-group G is called *residually*-$\mathcal{C}$ if G can be represented as a subdirect product of o-groups in $\mathcal{C}$. We shall denote the class of all residually-$\mathcal{C}$ l-groups by Res($\mathcal{C}$). To say that $G \in$ Res($\mathcal{C}$) is equivalent to the condition that for each $0 < x \in G$ there be a normal prime subgroup N of G, such that $x \notin N$ and $G/N \in \mathcal{C}$. An l-group G is *hyper*-$\mathcal{C}$ if it is residually-$\mathcal{C}$ and every l-homomorphic image of G is residually-$\mathcal{C}$. Hyp($\mathcal{C}$) is the class of all hyper-$\mathcal{C}$ l-groups. If $\mathcal{C}$ is the class of archimedean o-groups then "hyper-$\mathcal{C}$" coincides with the usual notion of hyper-archimedeaneity.

If $\mathcal{V}$ is a variety of representable l-groups and $\mathcal{C}$ is the class of o-groups in $\mathcal{V}$ then $\mathcal{V}$ = Res($\mathcal{C}$) = Hyp($\mathcal{C}$). Such a class of o-groups is said to be *full*. We remark here that for any class $\mathcal{C}$, the variety generated by $\mathcal{C}$ is obtained by taking all l-homomorphic images of residually-$\mathcal{C}$ l-groups. Thus, if $\mathrm{Res}(\mathcal{C}) = \mathrm{Hyp}(\mathcal{C})$ then $\mathcal{C}$ is a full class of o-groups.

A convex l-subgroup C of an l-group G is *regular* if C is maximal with respect to not containing some $0 \neq g \in G$; we also say that C is a *value* of g. An element $0 \neq g \in G$ is *special* if it has only one value; the value of a special element is also said to be *special*. G is a *finite-valued* l-group if every nonzero element has finitely many values. It is well known [4, Theorem 2.21] that G is finite valued if and only if every nonzero element $g \in G$ is a sum of finitely many special elements whose absolute values are pairwise disjoint.

If K is an l-ideal of G which is maximal without some element $0 \neq g \in G$ we say that K is *regular* (*among l-ideals*). An l-group is representable if and only if every regular l-ideal is a prime subgroup [4, Theorem 1.8] (regular convex l-subgroups are always prime). The meaning of the term *l-ideal value* is evident.

We now state the following:

1.1 **Proposition.** *If G is a finite-valued residually-$\mathcal{C}$ l-group, then G is hyper-$\mathcal{C}$.*

Proof. If $0 < g \in G$, write $g = g_1 + g_2 + \cdots + g_n$, where the g_i are pairwise disjoint special elements. If M is a regular l-ideal of G maximal without g, then some $g_j \notin M$. Suppose M' is an l-ideal that contains M properly; since $g \in M'$ it must be true that all the g_i are in M'. The conclusion is that M is an l-ideal value of g_j. Now g_j is special so that M is contained in Q, the unique value of g_j. The conjugates Q^x of Q form a chain and $\bigcap\{Q^x \mid x \in G\} = M$ [4, Theorem 1.8]; this proves that M is the only l-ideal value of g_j.

We have shown that all regular l-ideals are (unique) values of special elements, among l-ideals. Since $G \in \mathrm{Res}(\mathcal{C})$ there is a family of normal prime subgroups $\{N_\gamma \mid \gamma \in \Gamma\}$ with zero intersection such that $G/N_\gamma \in \mathcal{C}$, for each $\gamma \in \Gamma$. For any regular l-ideal M, choose a special element x of which M is *the* l-ideal value. There is a $\gamma_0 \in \Gamma$ such that $x \notin N_{\gamma_0}$, and hence $N_{\gamma_0} \subseteq M$; G/M is a quotient of G/N_{γ_0}, and so $G/M \in \mathcal{C}$.

Now if K is an l-ideal of G, it is the meet of regular l-ideals. But we have just shown that the factor of such an l-ideal in G is in $\mathcal{C}$; it follows that $G/K \in \mathrm{Res}(\mathcal{C})$. This proves G is hyper-$\mathcal{C}$, and we are done. □

Let us now consider some "new" examples of closed classes of o-groups.

(I) An o-group G is *c-archimedean* (*conjugate-archimedean*) if for each $0 < x \in G$ and $g \in G$ there is a positive integer n such that $nx \geq x^g$. The class of c-archimedean o-groups is closed. It is clear that an o-group G is c-archimedean if and only if every convex subgroup of G is normal.

(II) An l-group G is *weakly abelian* if, for each $0 < x \in G$ and $g \in G$, $2x \geq x^g$. The class of weakly abelian l-groups forms a variety, so that the weakly abelian o-groups are a full closed class.

If $\mathcal{C}$ is the class of c-archimedean o-groups we shall call an l-group in $\mathrm{Hyp}(\mathcal{C})$ *hyper c-archimedean.* It is evident that every weakly abelian l-group is hyper c-archimedean. The residually-$\mathcal{C}$ l-groups satisfy the condition that for each $0 < x \in G$ and $g \in G$ there is a positive integer n such that $nx \nleq x^g$; l-groups satisfying this last axiom shall be called *c-archimedean.* We then have the following characterization of hyper c-archimedean l-groups.

1.2 **Theorem.** *For a representable l-group G, the following are equivalent:*

(i) *G is hyper c-archimedean;*

(ii) *G is c-archimedean and every l-homomorphic image is c-archimedean.*

(iii) *If N is a prime of G then N is normal in G.*

(iv) *All convex l-subgroups of G are normal in G.*

(v) *If $0 < x \in G$, $g \in G$ there is a positive integer n such that $nx \geq x^g$.*

Proof. (i) $\longrightarrow$ (ii). If G is hyper c-archimedean it is certainly c-archimedean, and every quotient is a subdirect product of c-archimedean o-groups, and hence c-archimedean.

(ii) $\longrightarrow$ (iii). Representability guarantees that minimal primes are normal. If N is a minimal prime then G/N is c-archimedean. If M is any prime of G, it contains a minimal prime M' and M/M' is normal in G/M', whence M is normal in G.

(iii) $\longrightarrow$ (iv). Obvious since every convex l-subgroup is the meet of primes.

(iv) $\longrightarrow$ (v). Clear.

(iv) $\longrightarrow$ (i). If all the convex l-subgroups of G are normal, and K is an l-ideal of G every prime subgroup containing K is normal. Thus, G/K is a subdirect product of c-archimedean o-groups, and G is therefore hyper c-archimedean.

If $\mathfrak{R}$ denotes the set of regular convex l-subgroups of an l-group G, we call a subset $\mathcal{P}$ of $\mathfrak{R}$ *plenary* if $\bigcap \mathcal{P} = 0$ and $\mathcal{P}$ is a dual ideal in $\mathfrak{R}$; that is, $M \in \mathcal{P}$ and $M \subseteq N \in \mathfrak{R}$ imply that $N \in \mathcal{P}$.

1.3 **Theorem.** *Suppose $\mathcal{C}$ denotes the class of c-archimedean o-groups. Then $G \in \mathrm{Res}(\mathcal{C})$ if and only if there is a plenary subset of regular subgroups of G in which a nonzero element of G has the same values as any of its conjugates.*

Proof. Let us suppose first that G is residually-$\mathcal{C}$. We consider all regular subgroups M having the property that M is normal in G and G/M is a c-archimedean o-group; call the family of all such regular subgroups $\mathfrak{M}$. If $0 < x \in G$ then we can find a normal prime N such that $x \notin N$ and G/N is c-archimedean. Now N is contained in some value M of x, and it is clear (see Theorem 1.2) that $M \in \mathfrak{M}$; this proves that $\bigcap \mathfrak{M} = 0$. Since the class $\mathcal{C}$ is quotient closed $\mathfrak{M}$ is a dual ideal; we conclude that $\mathfrak{M}$ is a plenary subset. For each $M \in \mathfrak{M}$, $G/M \in \mathcal{C}$ so that it is obvious that any nonzero element of G has the same values in $\mathfrak{M}$ as any of its conjugates.

Conversely, if there is such a plenary subset $\mathcal{P}$, then any regular subgroup $M \in \mathcal{P}$ is normal in G. Since $\mathcal{P}$ is a dual ideal every convex subgroup of G/M is normal, which says that $G/M \in \mathcal{C}$. $\bigcap \mathcal{P} = 0$, which implies that G is a subdirect product of c-archimedean o-groups, and our proof is complete.

1.3.1 **Corollary.** *G is hyper c-archimedean if and only if for each $0 \neq x \in G$ and $g \in G$ all the values of x and x^g coincide.*

We now return to an arbitrary (closed) class $\mathcal{C}$. The next lemma will be extremely useful.

1.4 **Lemma.** *A representable l-group G is hyper-$\mathcal{C}$ if and only if $G/M \in \mathcal{C}$ for every regular l-ideal M of G.*

Proof. Suppose first that $G \in \mathrm{Hyp}(\mathcal{C})$. An o-group with a minimal l-ideal is subdirectly irreducible. If M is a regular l-ideal then G/M is such an o-group; since $G \in \mathrm{Hyp}(\mathcal{C})$, G/M is residually-$\mathcal{C}$ and hence in $\mathcal{C}$.

Conversely, if $G/M \in \mathcal{C}$ for every regular l-ideal M, consider an arbitrary l-ideal K of G; K is the meet of regular l-ideals, and we conclude that G/K is residually-$\mathcal{C}$. This proves that G is hyper-$\mathcal{C}$ and we are done. □

Remark. Suppose H is an o-group which is residually-$\mathcal{C}$. Then H need not be in $\mathcal{C}$ (we shall give examples later). Thus if G is an l-group in $\mathrm{Hyp}(\mathcal{C})$ and N is a normal prime subgroup, G/N need not be in $\mathcal{C}$ although it is residually-$\mathcal{C}$.

1.5 **Proposition.** (i) *If $G \in \mathrm{Hyp}(\mathcal{C})$ and H is an l-subgroup of G, then H is hyper-$\mathcal{C}$.*

(ii) $\mathrm{Hyp}(\mathcal{C})$ *is closed under quotients and cardinal sums.*

Proof. (i) Suppose N is a regular l-ideal of H; there is an l-ideal N' of G such that $N = N' \cap H$. Then $H/N \simeq H/(N' \cap H) \simeq N' + H/N'$, and $N' + H/N'$ is an o-subgroup of $G/N' \in \mathrm{Res}(\mathcal{C})$; hence H/N is also residually-$\mathcal{C}$—and therefore in $\mathcal{C}$, since it is subdirectly irreducible—since $\mathrm{Res}(\mathcal{C})$ is l-subgroup closed. By Lemma 1.4, we conclude that $H \in \mathrm{Hyp}(\mathcal{C})$.

(ii) It is trivial that $\mathrm{Hyp}(\mathcal{C})$ is closed under l-homomorphic images. As for cardinal sums, suppose $G = \boxplus \{G_\lambda \mid \lambda \in \Lambda\}$ and each G_λ is hyper-$\mathcal{C}$. If N is a regular l-ideal of G then, since N is prime, it contains all the G_λ save one, say G_{λ_0}. Then $G/N \simeq G_{\lambda_0}/(N \cap G_{\lambda_0})$, and the latter is in $\mathcal{C}$ since $N \cap G_{\lambda_0}$ is a regular l-ideal of G_{λ_0}. Again G is hyper-$\mathcal{C}$ by Lemma 1.4. □

We now turn to classes which have the following additional property: if G is an o-group having a family of convex subgroups $\{G_\gamma \mid \gamma \in \Gamma\}$ such that $G = \bigcup G_\gamma$ and each $G_\gamma \in \mathcal{C}$ then $G \in \mathcal{C}$. This is clearly a local property of sorts; if $\mathcal{C}$ has this property we say $\mathcal{C}$ is *locally closed with respect to convex subgroups*. Actually, there will be no confusion in suppressing the qualification "with respect to convex subgroups", and we shall do so throughout. Further, the convex subgroups of an o-group lie on a chain, so we are dealing with local closure relative to *directed* families of convex subgroups.

We can now state and prove the main result in this section.

1.6 Theorem. *Suppose $\mathcal{C}$ is a locally closed class of o-groups, and G is a representable l-group. There is an l-ideal $\mathcal{C}(G)$ of G which is hyper-$\mathcal{C}$ and contains every hyper-$\mathcal{C}$ convex l-subgroup of G. $\mathcal{C}(G)$ is in fact a characteristic subgroup; (that is, it is left invariant by any l-automorphism of G).*

Proof. Consider the family $\mathcal{H}$ of hyper-$\mathcal{C}$ convex l-subgroups of G. We will show $\mathcal{H}$ is a complete sublattice of the lattice of convex l-subgroups. By 1.5(i), $\mathcal{H}$ is certainly closed under intersections, so it suffices to show that the join of hyper-$\mathcal{C}$ convex l-subgroups is hyper-$\mathcal{C}$. Suppose then that $\{K_\lambda \mid \lambda \in \Lambda\}$ is a subfamily of $\mathcal{H}$, and let $K = \bigvee K_\lambda$. In view of representability and a result of Wolfenstein in [10] any two convex l-subgroups of G commute. Let N be a regular l-ideal of K; the convex l-subgroup generated by N and K_λ is $N + K_\lambda$, and we have

$$K/N \simeq \bigvee \{N + K_\lambda/N \mid \lambda \in \Lambda\}.$$

Each $N + K_\lambda/N \simeq K_\lambda/N \cap K_\lambda$, and each $K_\lambda/N \cap K_\lambda \in \mathcal{C}$ since $K_\lambda \in \mathcal{H}$ (using Lemma 1.4 once more). So K/N is the join of a chain of convex subgroups in $\mathcal{C}$ and is therefore itself in $\mathcal{C}$. It follows that $K \in \mathcal{H}$.

The theorem now ensues in its entirety since $\mathcal{C}(G) = \bigvee \mathcal{H}$ is obviously characteristic, and hence an l-ideal.

1.7 Theorem. *Once again let $\mathcal{C}$ be a locally closed class, and G be a representable l-group. Then*

(i) $A \cap \mathcal{C}(G) = \mathcal{C}(A)$, *for every convex l-subgroup A of G;*

(ii) $\mathcal{C}(\mathcal{C}(G)) = \mathcal{C}(G)$;

(iii) $\mathcal{C}(A + B) = \mathcal{C}(A) + \mathcal{C}(B)$, *for any convex l-subgroups A and B;*

(iv) $\mathcal{C}(A \cap B) = \mathcal{C}(A) \cap \mathcal{C}(B)$, *for any convex l-subgroups A and B.*

Proof. (i) Clearly $A \cap \mathcal{C}(G) \subseteq \mathcal{C}(A)$; on the other hand, if K is a hyper-$\mathcal{C}$ convex l-subgroup of A, it is also a convex l-subgroup of G, whence $K \subseteq A \cap \mathcal{C}(G)$.

(ii) Trivial.

(iii) By the proof of 1.6, $\mathcal{C}(A) + \mathcal{C}(B) \subseteq \mathcal{C}(A + B)$. If C is a hyper-$\mathcal{C}$ convex l-subgroup of $A + B$ then $C = (C \cap A) + (C \cap B)$, since the lattice of convex l-subgroups is distributive [4, Theorem 1.4]. But $(C \cap A) + (C \cap B) \subseteq \mathcal{C}(A) + \mathcal{C}(B)$ and the desired equality follows.

We leave (iv) to the reader.

We call $\mathcal{C}(G)$ the *hyper-$\mathcal{C}$-kernel* of an l-group G. In view of what is absent in 1.7 one immediately wonders what can be said about $G/\mathcal{C}(G)$. This l-group is again representable, so one may speak of its hyper-$\mathcal{C}$-kernel; there is a characteristic l-ideal of G, $\mathcal{C}^2(G)$, so that $\mathcal{C}^2(G)/\mathcal{C}(G) = \mathcal{C}(G/\mathcal{C}(G))$. Inductively, we define, for an ordinal σ, $\mathcal{C}^{\sigma+1}(G)$ to be that characteristic l-ideal of G satisfying $\mathcal{C}^{\sigma+1}(G)/\mathcal{C}^{\sigma}(G) = \mathcal{C}(G/\mathcal{C}^{\sigma}(G))$. If τ is a limit ordinal let $\mathcal{C}^{\tau}(G) = \bigcup\{\mathcal{C}^{\sigma}(G) \mid \sigma < \tau\}$. The chain $\mathcal{C}(G) = \mathcal{C}^1(G) \subseteq \mathcal{C}^2(G) \subseteq \cdots \subseteq \mathcal{C}^{\sigma}(G) \subseteq \cdots$ is called the *hyper-$\mathcal{C}$-kernel* sequence of the l-group G.

For the remainder of this section we shall look at some examples. All examples of closed classes of o-groups we have given so far are locally closed. (The full classes are locally closed because varieties are closed under direct limits.) At the end of the section we shall point out some examples that are not locally closed.

Suppose $\mathfrak{V}$ and $\mathfrak{W}$ are varieties of l-groups; the *product variety* $\mathfrak{V} \cdot \mathfrak{W}$ is the variety of all l-groups G having an l-ideal A in $\mathfrak{V}$ such that $G/A \in \mathfrak{W}$. One easily checks that $\mathfrak{V} \cdot \mathfrak{W}$ is indeed a variety, and that this product is associative. (The proofs of these two facts are identical to the ones for groups; see [8].) So now let $\mathcal{C}$ be the full class of o-groups of a variety of representable l-groups $\mathfrak{V}$. Then $\mathcal{C}^n(G) = G$ for a positive integer n if and only if $G \in \mathfrak{V}^n = \mathfrak{V} \cdot \mathfrak{V} \cdot \cdots \cdot \mathfrak{V}$ (n times).

Let $\mathfrak{A}$ denote the variety of abelian l-groups. It is well known that a free group admits total orders (sse [6]); obviously the hyper-$\mathfrak{A}$-kernel of such a free group is zero. Call a representable l-group *anti-abelian* if $\mathfrak{A}(G) = 0$. The class of anti-abelian l-groups is closed under taking convex l-subgroups (1.7(i)), and subdirect products. The last assertion follows from the more general fact:

1.8 **Proposition.** *If $\mathcal{C}$ is a locally closed class and α is an l-homomorphism of the representable l-group G onto H then* $[\mathcal{C}(G)]\alpha \subseteq \mathcal{C}(H)$.

Proof. $[\mathcal{C}(G)]\alpha$ is a convex l-subgroup of H, and is certainly hyper-$\mathcal{C}$; the desired inclusion follows. □

If $\alpha: G \longrightarrow H$ is as in Proposition 1.8, and H is anti-abelian, then $\mathfrak{A}(G) \subseteq \mathrm{Ker}(\alpha)$. So if $\sigma: G \longrightarrow \Pi G_i$ is a subdirect representation of the l-group G by anti-abelian l-groups then G is anti-abelian. It follows then from Conrad's representation for free representable l-groups [4, Chapter 6] that a free representable l-group is anti-abelian. We also see that in Proposition 1.8 equality does not hold in general; in fact, every abelian l-group is the image of an anti-abelian l-group.

Let us give two examples to show what relationship the center $Z(G)$ of an l-group G might have with $\mathfrak{A}(G)$. Let $G = \mathbf{Z} \text{ wr } \mathbf{Z}$, the restricted wreath product of $\mathbf{Z}$, the additive group of integers, with itself; we declare $(k; \cdots, a_n, \cdots) > 0$ if $k > 0$, or $k = 0$ and the first nonzero a_n is positive. G is an o-group with trivial center, yet $\mathfrak{A}(G) = \{(k; \cdots, a_n, \cdots) \in G \mid k = 0\}$.

On the other hand, suppose G is the group of triangular 2 by 2 matrices

$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with rational entries, and such that $a > 0$, $c > 0$, all under matrix-multiplication. If 1 denotes the identity matrix set

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \geq 1$$

if $a > 1$, or $a = 1$ and $c > 1$, or $a = c = 1$ and $b \geq 0$. Then G is an o-group whose center consists of all diagonal matrices, yet

$$\mathfrak{A}(G) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \text{ rational} \right\}.$$

Notice that $Z(G)$ is not convex, and $Z(G) \cap \mathfrak{A}(G) = 1$.

Let us now turn to the situation with archimedean l-groups. For now let $\mathfrak{A}_1$ denote the class of archimedean o-groups. Let G be a representable l-group, and $0 < g \in G$; $g \in \mathfrak{A}_1(G)$ if and only if $G(g)$ is hyper-archimedean. Suppose $0 < g \in \mathfrak{A}_1(G)$; if N is a minimal prime of G with $g \notin N$ then $N \cap G(g)$ is a normal maximal convex l-subgroup of $G(g)$ [4, Theorem 2.4]. Moreover, there is a one-to-one correspondence between the convex l-subgroups of G that do not contain g, and the proper convex l-subgroups of $G(g)$ (this correspondence is effected by intersecting with $G(g)$; see [4, Theorem 1.14]). Thus N is maximal with respect to missing g, and is therefore regular. Conversely suppose every minimal prime of G missing g is a value of g. For each such prime M, $M \cap G(g)$ is a maximal l-ideal, and clearly the primes of $G(g)$ are trivially ordered, whence $G(g)$ is hyper-archimedean. We have proved

1.9 **Proposition.** *Let G be a representable l-group; then $0 < g \in \mathfrak{A}_1(G)$ if and only if every minimal prime of G without g is a value of g.*

1.9.1 **Corollary.** *For a representable l-group G, $\mathfrak{A}_1(G)$ is the intersection of all primes of G which are not minimal.*

1.9.2 **Corollary.** *If G is finite valued (and representable) then $\mathfrak{A}_1(G) = 0$ if and only if G has no regular minimal prime subgroups.*

An element $0 < s$ of an l-group G is *basic* if the set $\{g \in G \mid 0 \leq g \leq s\}$ is a chain. A *basis* for an l-group is a maximal set of pairwise disjoint elements each of which is basic. If G is archimedean then $\mathfrak{A}_1(G)$ contains every basic element; there are, however, hyper-archimedean l-groups with no basic elements.

The element $s > 0$ in a representable l-group G is *singular* if $0 \leq g \leq s$ implies that $g \wedge (s - g) = 0$.

1.9.3. **Corollary.** *If $0 < s \in G$ is singular then $s \in \mathfrak{A}_1(G)$.*

Proof. By a result of Conrad and McAlister [5, Lemma 4.5] every value of a singular element is a minimal prime.

However, an l-group may have a nonzero hyper-archimedean kernel without having any singular elements. If $G = \Pi\{\mathbf{R}_\lambda \mid \lambda \in \Lambda\}$ (Λ an infinite set), then $\mathfrak{A}_\pi(G)$ is the cardinal sum of the $\mathbf{R}_\lambda$, yet G has no singular elements.(2) On the other hand, if $H = \Pi\{\mathbf{Z}_\lambda \mid \lambda \in \Lambda\}$ then $\mathfrak{A}_\pi(H)$ is the l-ideal of all bounded integral sequences. Notice that $H \cap \mathfrak{A}_\pi(G) \neq \mathfrak{A}_\pi(H)$.

For certain subdirect products of integers we can describe $\mathfrak{A}_\pi(G)$.

1.10 **Proposition.** *Suppose G is a subdirect product of integers, say $G \subseteq \Pi\{\mathbf{Z}_\lambda \mid \lambda \in \Lambda\}$; suppose also that G contains a bounded weak order unit. Then $\mathfrak{A}_\pi(G)$ consists of the bounded elements of G.*

(Recall that $0 < e \in G$ is a *weak order unit* if $e \wedge g > 0$, whenever $0 < g \in G$.)

Proof. Clearly every bounded element is in $\mathfrak{A}_\pi(G)$, for the bounded elements form an l-ideal; and if $a, b \in G$ are bounded there is a positive integer n such that $n[\min\{|a_\lambda| \mid \lambda \in \text{support}(a)\}] \geq \sup\{|b_\lambda| \mid \lambda \in \text{support}(b)\}$.(3)

Suppose conversely that $0 < x \in \mathfrak{A}_\pi(G)$; let u be a bounded weak order unit in $\mathfrak{A}_\pi(G)$. Then $\mathfrak{A}_\pi(G) = G(u) \boxplus P$, where P is the polar of u in $\mathfrak{A}_\pi(G)$ [4, Theorem 2.4]. But $P = 0$, so $mu \geq x$, for a suitable positive integer m; this means x is bounded, and the proof is complete. $\square$

We now consider the hyper-archimedean kernel sequence. Obviously, G is an extension of a hyper-archimedean l-group by another if and only if $\mathfrak{A}_\pi^2(G) = G$. (In fact, for any locally closed class $\mathcal{C}$, $G = \mathcal{C}^2(G)$ if and only if G is an extension of a hyper-$\mathcal{C}$ l-group by another.) For such l-groups we get

1.11 **Proposition.** *If $G = \mathfrak{A}_\pi^2(G)$ then each root in the root system of primes of G has at most length 2 (that is, every prime is either minimal or maximal). Further if N is a prime which is not maximal then there is an element $0 < a \in \mathfrak{A}_\pi(G)$ whose value is N.*

Conversely, suppose G is a representable l-group with the property that a prime of G is either minimal or maximal. Let A be the intersection of all prime subgroups of G that contain a prime properly; then $\mathfrak{A}_\pi(G) = A$, and G/A is hyper-archimedean if for each minimal prime N of G that is not maximal there is an element $0 < a \in A$ whose value is N.

Proof. Suppose $G = \mathfrak{A}_\pi^2(G)$, and N is a prime of G. If $N \supseteq \mathfrak{A}_\pi(G)$ then $N/\mathfrak{A}_\pi(G)$ is a maximal prime of $G/\mathfrak{A}_\pi(G)$; thus N is maximal in G. If $N \not\supseteq \mathfrak{A}_\pi(G)$ then $N \cap \mathfrak{A}_\pi(G)$ is a maximal prime of $\mathfrak{A}_\pi(G)$, which implies that N is maximal with respect to not containing $\mathfrak{A}_\pi(G)$. Hence N is regular, and its cover is G itself or a maximal prime subgroup containing $\mathfrak{A}_\pi(G)$. Further $N \cap \mathfrak{A}_\pi(G)$ is also a minimal prime of $\mathfrak{A}_\pi(G)$, hence N is a minimal prime of G.

(2) $\mathbf{R}$ denotes the additive group of real numbers with its natural ordering.

(3) See Theorem 2.4 of [4].

If N is not maximal it is minimal and $\mathfrak{A}_r(G) \not\subseteq N$, so we may select $0 < a \in \mathfrak{A}_r(G)\backslash N$; by Proposition 1.9, N is a value of a.

The converse of this proposition is clear ($A = \mathfrak{A}_r(G)$ by Corollary 1.9.1).

1.11.1 Corollary. *If $G = \mathfrak{A}_r^n(G)$ then a root in the root system of primes of G has at most length n. In particular every prime is regular.*

1.11.2 Corollary. *Suppose G is finite valued. Then $G = \mathfrak{A}_r^n(G)$ and $\mathfrak{A}_r^{n-1}(G) \subset G$ if and only if the maximum length of a root of primes is exactly n.*

Finally, since the hyper-$\mathcal{C}$-kernel is always characteristic, and the free abelian l-group is characteristically simple [1], we have

1.12 Proposition. *If G is a free abelian l-group then $\mathfrak{A}_r(G) = 0$.*

The class $\mathcal{Z}$ consisting of cyclic o-groups is certainly locally closed. The residually-$\mathcal{Z}$ l-groups are the subdirect products of integers, and the hyper-$\mathcal{Z}$ l-groups are the l-groups G in which every prime subgroup has a cyclic factor in G. The hyper-$\mathcal{Z}$-kernel of an l-group H consists of those elements of H whose values are minimal primes with cyclic factor in their respective covers.

To conclude the section let us list some examples of closed classes of o-groups which are not locally closed.

(a) All o-groups admitting a finite central chain of (normal) convex subgroups. This class is closed under taking subgroups and quotients since the convex subgroups of an o-group form a chain.

(b) All o-groups whose chain of convex subgroups is finite.

The reader may think up various variations on the theme in these examples, such as: all o-groups with the ascending chain condition on convex subgroups, all o-groups with a terminating descending central chain, etc.

2. Para-$\mathcal{C}$ l-groups. We exhibited in the residually-$\mathcal{C}$ and hyper-$\mathcal{C}$ l-groups associated with a class $\mathcal{C}$ of o-groups some archimedean-like behaviour. In this section we try to generalize the notion of archimedeaneity itself.

Let $\mathcal{C}$ be a closed class of o-groups; an l-group G is said to be *para-$\mathcal{C}$* if there is a directed system of convex l-subgroups of G each of which is residually-$\mathcal{C}$, and whose union is G.

When $\mathcal{C} = \mathfrak{A}_r$, the class of archimedean o-groups, we have

2.1 Proposition. *The para-$\mathfrak{A}_r$ l-groups are precisely the archimedean l-groups.*

Proof. Certainly an l-group in which every principal convex l-subgroup is archimedean is itself archimedean (see Proposition 2.2(i) below). On the other hand, if G is an archimedean l-group and $0 < x \in G$ then $G(x)$ is an archimedean l-group with a strong order unit. Thus $G(x)$ is a subdirect product of reals, and of course G is the union of the $G(x)$.

The following proposition lists some of the basic properties of Para $(\mathcal{C})$, the class of all para-$\mathcal{C}$ l-groups.

2.2 **Proposition.** (i) *An l-group is para-$\mathcal{C}$ if and only if $G(x)$ is residually-$\mathcal{C}$, for each $0 < x \in G$.*

(ii) Para $(\mathcal{C})$ *is closed under taking l-subgroups.*

(iii) Para $(\mathcal{C})$ *is residually closed.*

(iv) Hyp $(\mathcal{C}) \subseteq$ Res $(\mathcal{C}) \subseteq$ Para $(\mathcal{C}) \subseteq$ Var $(\mathcal{C}) \equiv$ *the variety of l-groups generated by $\mathcal{C}$. If $\mathcal{C}$ is a full class then equality holds throughout. (Note: in general* Var $(\mathcal{C})$ *consists of all l-homomorphic images of residually-$\mathcal{C}$ l-groups.)*

(v) *A para-$\mathcal{C}$ l-group with a strong order unit is residually-$\mathcal{C}$.*

Proof. (i) Clear, since Res $(\mathcal{C})$ is closed under taking l-subgroups.

(ii) Suppose H is an l-subgroup of $G \in$ Para (C), and $0 < x \in H$. $H \cap G(x) = H(x)$, and since $G(x)$ is residually-$\mathcal{C}$ so is $H(x)$; from (i) we conclude H is para-$\mathcal{C}$.

(iii) By (ii) just proved, it suffices to show that a cardinal product of para-$\mathcal{C}$ l-groups is para-$\mathcal{C}$. Suppose $G = \Pi\{G_\gamma \mid \gamma \in \Gamma\}$ with each G_γ in Para $(\mathcal{C})$, and take $0 < x \in G$. Under the γth projection $G(x)$ is mapped onto $G_\gamma(x_\gamma)$, and so $G(x)$ is isomorphic to a subdirect product of the $G_\gamma(x_\gamma)$ $(\gamma \in \Gamma)$. But each $G_\gamma(x_\gamma)$ is residually-$\mathcal{C}$, whence $G(x)$ is residually-$\mathcal{C}$. By (i), $G \in$ Para $(\mathcal{C})$.

(iv) and (v) are obvious and are left to the reader. □

For our next important result we need a lemma.

2.3 **Lemma.** *Suppose G is residually-$\mathcal{C}$ and $0 < a \in G$; then a', the polar of a, is the meet of normal prime subgroups N of G having the property that $a \notin N$ and $G/N \in \mathcal{C}$.*

Proof. Certainly a' is contained in any such prime. On the other hand, if $b \wedge a > 0$, then since $G \in$ Res $(\mathcal{C})$ there is a normal prime subgroup N of G such that $G/N \in \mathcal{C}$ that misses both a and b.

2.4 **Theorem.** *Suppose G is para-$\mathcal{C}$ and P is a polar subgroup of G; then $G/P \in$* Para $(\mathcal{C})$. *If, moreover, $G \in$* Res $(\mathcal{C})$ *then G/P is also in* Res $(\mathcal{C})$.

Proof. Since a polar of an l-group is the meet of principal polars, and Para $(\mathcal{C})$ is residually closed, it suffices to let $P = a'$ for some $0 < a \in G$. We suppose first that G is in fact residually-$\mathcal{C}$; it is immediate from Lemma 2.3 that $G/a' \in$ Res $(\mathcal{C})$. We weaken the hypothesis and take G to be para-$\mathcal{C}$. For each $0 < x \in G$ such that $a \in G(x)$ we have that $G(x)/G(x) \cap a' \in$ Res $(\mathcal{C})$, since $G(x) \in$ Res $(\mathcal{C})$ and $G(x) \cap a'$ is the polar of a in $G(x)$. Now $G(x) + a'/a' \simeq G(x)/G(x) \cap a'$, and so $G(x) + a'/a'$ is residually-$\mathcal{C}$; G is the union of all the $G(x) + a'$, allowing us to deduce that $G/a' \in$ Para $(\mathcal{C})$. This completes the proof of the theorem.

Under some additional hypotheses we can obtain more information about Para$(\mathcal{C})$.

2.5 **Proposition.** *Suppose* $\mathcal{C}$ *is a locally closed class and* G *is a finite valued para-*$\mathcal{C}$ *l-group; then* G *is hyper-*$\mathcal{C}$*.*

Proof. If G is a finite-valued para-$\mathcal{C}$ l-group then for each $0 < g \in G$ $G(g)$ is finite valued and residually-$\mathcal{C}$, and hence hyper-$\mathcal{C}$ by 1.1. G is the direct limit of hyper-$\mathcal{C}$ convex l-subgroups, in other words $G = \mathcal{C}(G)$, and G itself is hyper-$\mathcal{C}$. □

We have already indicated that a residually-$\mathcal{C}$ o-group need not be in $\mathcal{C}$. Suppose, for example, that $\mathcal{C}$ consists of all o-groups whose chain of convex subgroups satisfies the descending chain condition. $\mathcal{C}$ is subgroup and quotient closed, and is in fact locally closed. But if $G = \bigoplus_{n=-\infty}^{\infty} \mathbf{Z}_n$, where $\mathbf{Z}_n = \mathbf{Z}$ for each integer n, and the ordering is lexicographic from the left, then $G \notin \mathcal{C}$, yet it has enough quotients in $\mathcal{C}$ to make it residually-$\mathcal{C}$. If $\mathcal{C}$ does have the property that residually-$\mathcal{C}$ o-groups are in $\mathcal{C}$, we say that $\mathcal{C}$ is *residually closed.* The following classes are all residually closed: the archimedean o-groups, the c-archimedean o-groups, any full class of o-groups.

2.6 **Proposition.** *If* $\mathcal{C}$ *is locally closed and residually closed we have*

(1) *a para-*$\mathcal{C}$ *o-group is in* $\mathcal{C}$;

(2) G *is hyper-*$\mathcal{C}$ *if and only if* G *is para-*$\mathcal{C}$ *and every l-homomorphic image of* G *is para-*$\mathcal{C}$*.*

Proof. (1) If G is a para-$\mathcal{C}$ o-group each $G(x)$ is a residually-$\mathcal{C}$ p-group, and so $G(x) \in \mathcal{C}$. In view of the local closure of $\mathcal{C}$ we conclude that $G \in \mathcal{C}$.

(2) If $G \in \mathrm{Hyp}(\mathcal{C})$ it is clearly in Para$(\mathcal{C})$; moreover each l-homomorphic image of G is residually-$\mathcal{C}$ and hence also para-$\mathcal{C}$. Conversely, suppose $G \in$ Para$(\mathcal{C})$ and each l-homomorphic image of G is para-$\mathcal{C}$. For each regular l-ideal M of G, $G/M \in$ Para$(\mathcal{C})$ and is an o-group in view of the representability of G. By (1) $G/M \in \mathcal{C}$; Lemma 1.4 now guarantees that G is hyper-$\mathcal{C}$.

2.7 **Proposition.** *Suppose* $\mathcal{C}$ *is locally closed and residually closed. Let* $G \in$ Para$(\mathcal{C})$ *and* $0 < a \in G$ *be a basic element. Then* $G/a' \in \mathcal{C}$; *consequently, if* G *has a basis it is residually-*$\mathcal{C}$*.*

Proof. G/a' is para-$\mathcal{C}$ by 2.4; since it is an o-group [4, Theorem 2.2] it is in $\mathcal{C}$ by 2.6(1). If G has a basis $\{s_\lambda | \lambda \in \Lambda\}$ then G is subdirectly representable by a product of the G/s_λ'; clearly then $G \in$ Res$(\mathcal{C})$. □

Consider now the special case of the class of c-archimedean o-groups. We have already seen that the subdirect products of c-archimedean o-groups are characterized by the condition that there is a plenary subset in which any nonzero element has the same values as any of its conjugates (Theorem 1.3). We can characterize the representable c-archimedean l-groups as follows.

2.8 **Theorem.** *Suppose G is a representable l-group; G is c-archimedean if and only if for each $0 < x \in G$ and $g \in G$ there is a value of x which is also a vlaue of x^g.*

Proof. The sufficiency is clear. As for the necessity, suppose no value of $x > 0$ coincides with a value of x^g. In view of representability, a value of x either contains a value of x^g properly, or is properly contained in one. Consider $y = (x^g - x) \vee 0$; the values of y are precisely those values of x^g which exceed a value of x. (Since G is c-archimedean there is at least one such.) But then the values of y^g are precisely the values of x^{2g} which exceed a value of x^g. Consequently, every value of y is properly contained in a value of y^g, that is, $ny \leq y^g$ for all $n = 1, 2, \cdots$, which contradicts the assumption that G is c-archimedean.

We conclude therefore that x must share some value with x^g. □

We would like to know whether a similar condition characterizes the para-c-archimedean l-groups. The following condition seems stronger than that of Theorem 2.8 yet weaker than that of Theorem 1.3: For any $0 < x \in G$ there is a value of x which is a value of any conjugate of x. We get the following containments:

$$\mathrm{Res}\,(\mathcal{C}) \subseteq \mathrm{Para}\,(\mathcal{C}) \subseteq \mathcal{C}^* \quad \text{and} \quad \mathrm{Res}\,(\mathcal{C}) \subseteq \mathfrak{D} \subseteq \mathcal{C}^*,$$

where $\mathcal{C}$ denotes the class of c-archimedean o-groups, $\mathcal{C}^*$ the class of all representable c-archimedean l-groups, and $\mathfrak{D}$ represents the class of all l-groups satisfying the last condition set forth above. It seems unlikely that equality will hold in any of these cases, but we lack counterexamples. $\mathrm{Para}\,(\mathcal{C})$ and $\mathfrak{D}$ also appear to be incomparable.

We do get the following partial result.

2.9 **Proposition.** *A finite-valued, representable, c-archimedean l-group G is hyper c-archimedean.*

Proof. Let M be a regular subgroup of G; since M is special [4, Corollary II to Lemma 2.20] there is a special element $0 < a \in G$ whose value is M. For any $g \in G$ the value of a^g is M^g, and M is comparable to M^g. Suppose $M \subset M^g$; then $na \leq a^g$ for each positive integer n, which contradicts the c-archimedeaneity of G. By symmetry $M^g \not\subset M$ and so $M = M^g$; the conclusion is that M is normal in G.

But if every regular subgroup is normal in G, so is every convex l-subgroup; by 1.2 we have that G is hyper c-archimedean.

A similar argument gives us

2.10 **Proposition.** *Suppose G is representable and has the property that every positive element $g \in G$ is the join (possibly infinite) of pairwise disjoint* *special elements. If G is c-archimedean it is a subdirect product of c-archimedean o-groups.*

Proof. The condition in the proposition is equivalent to the assertion that the set of special regular subgroups is plenary [3, Theorem 2.1]. Using the argument of 2.9 we know then that G/M is a c-archimedean o-group for every special regular subgroup M. This suffices to prove what we claim. $\square$

Another open question here is: what (elementwise) condition characterizes the para-$\mathcal{Z}$ l-groups, where $\mathcal{Z}$ is the class of cyclic o-groups? It seems to be quite a task to find a para-$\mathcal{Z}$ l-group which is not a subdirect product of integers.

In this connection we could pose a more general question: for any class $\mathcal{C}$, when is $\mathrm{Res}(\mathcal{C})$ a proper subclass of $\mathrm{Para}(\mathcal{C})$? If C is a full class we know that $\mathrm{Res}(\mathcal{C}) = \mathrm{Para}(\mathcal{C})$; is the converse true?

3. **Radicals and universal algebra.** It is well known that any variety of universal algebras has free algebras (see [2]). In particular, any variety of l-groups has free l-groups. We should like to describe these free l-groups when the variety is obtained from a closed class $\mathcal{C}$. $\mathrm{Var}(\mathcal{C})$ will denote the variety generated by $\mathcal{C}$; recall $\mathcal{C} \subset \mathrm{Hyp}(\mathcal{C}) \subseteq \mathrm{Res}(\mathcal{C}) \subseteq \mathrm{Para}(\mathcal{C}) \subseteq \mathrm{Var}(\mathcal{C})$.

We introduce the following auxiliary concept; given a representable l-group, G, let

$$R_{\mathcal{C}}(G) = \langle \{0 < g \in G \mid g \text{ has no } l\text{-ideal values } N \text{ with } G/N \in \mathcal{C}\} \rangle.$$

3.1 **Lemma.** *$R_{\mathcal{C}}(G)$ is an l-ideal of G.*

Proof. Let $R = \{0 < g \in G \mid g$ has no l-ideal values N with $G/N \in \mathcal{C}\}$. We show R is a normal convex subsemigroup of G. Normality is obvious; as for convexity, suppose $0 < h < g \in R$; if M is an l-ideal value of h such that G/M is in $\mathcal{C}$, then since $g \notin M$ there is an l-ideal value of g, say N, containing M. But G/N is a quotient of G/M and therefore in $\mathcal{C}$; consequently $h \in R$ since g has no such values.

Next, if $a, b \in R$ and N is an l-ideal value of $a + b$ with $G/N \in \mathcal{C}$, then N is an l-ideal value of a or b, a contradiction. Hence $a + b \in R$.

This suffices to prove the lemma.

3.2 **Lemma.** *$R_{\mathcal{C}}(G)$ is the meet of all the regular l-ideals whose factors in G are members of $\mathcal{C}$.*

Proof. Since an element outside $R_{\mathcal{C}}(G)$ does have a value whose factor in G is in $\mathcal{C}$, it is sufficient here to prove that every such regular l-ideal contains $R_{\mathcal{C}}(G)$. Suppose therefore that N is a regular l-ideal of G such that $G/N \in \mathcal{C}$. If $0 < a \in R_{\mathcal{C}}(G)\backslash N$ then a has an l-ideal value M containing N; as before $G/M \in \mathcal{C}$ which is contrary to assumption. It follows then that $R_{\mathcal{C}}(G) \subseteq N$, and we are done.

3.2.1 **Corollary.** *$G/R_{\mathcal{C}}(G)$ is a residually-$\mathcal{C}$ l-group. G is residually-$\mathcal{C}$ if and only if $R_{\mathcal{C}}(G) = 0$.*

Corollary 3.2.1 justifies our calling $R_{\mathcal{C}}(G)$ the *$\mathcal{C}$-radical* of G.

3.3 **Lemma.** *Suppose G and H are representable l-groups and ϕ is an l-homomorphism from G onto H. Then $[R_{\mathcal{C}}(G)]\phi \subseteq R_{\mathcal{C}}(H)$.*

Proof. We may suppose $H = G/K$, for a suitable l-ideal K of G. Suppose $0 < g \in R_{\mathcal{C}}(G)\setminus K$ and $D = N/K$ is an l-ideal value of $g + K$; then N is an l-ideal value of g, and $H/D \simeq G/N$, so that H/D is not in $\mathcal{C}$. Thus $g + K \in R_{\mathcal{C}}(H)$.

3.4 **Theorem.** *For any representable l-group G, $G/R_{\mathcal{C}}(G)$ is residually-$\mathcal{C}$. Moreover, if H is a residually-$\mathcal{C}$ l-homomorphic image of G then H is an l-homomorphic image of $G/R_{\mathcal{C}}(G)$.*

Proof. The first statement is Corollary 3.2.1. If $\alpha: G \longrightarrow H$ is an l-homomorphism onto $H \in \mathrm{Res}(\mathcal{C})$ then, by Lemma 3.2, $R_{\mathcal{C}}(G) \subseteq \mathrm{Ker}(\alpha)$, and so α factors through $G/R_{\mathcal{C}}(G)$.

3.4.1 **Corollary.** *If $\mathcal{C}$ is a full class then the $\mathcal{C}$-radical of an l-group G is the "word" l-ideal of G determined by* $\mathrm{Var}(\mathcal{C})$; *$R_{\mathcal{C}}(G)$ is therefore a fully invariant l-ideal (it is left invariant by all l-endomorphisms).*

We are now ready to prove the main theorem in this section.

3.5 **Theorem.** *For any set X the free l-group in* $\mathrm{Var}(\mathcal{C})$ *over X is residually-$\mathcal{C}$.*

Proof. Let $\alpha: X \longrightarrow G$ be a mapping into the residually-$\mathcal{C}$ l-group G. $\mathrm{F}(X)$ denotes the free representable l-group on X. There is a unique extension of α to an l-homomorphism β of $\mathrm{F}(X)$ into G; we may assume without loss of generality that β is onto G. If we set $F_{\mathcal{C}}(X) = \mathrm{F}(X)/R_{\mathcal{C}}(\mathrm{F}(X))$ then by Theorem 3.4 there is a unique l-homomorphism $\alpha^*: F_{\mathcal{C}}(X) \longrightarrow G$ such that $\eta\alpha^* = \beta$ (η is the natural map of $\mathrm{F}(X)$ onto $F_{\mathcal{C}}(X)$). It is clear that α determines α^* uniquely, and that the restriction of η to X is one-to-one, since α can be chosen one-to-one for a suitable $G \in \mathrm{Res}(\mathcal{C})$. This proves that $F_{\mathcal{C}}(X)$ is free over X for $\mathrm{Res}(\mathcal{C})$.

However, if $H \in \mathrm{Var}(\mathcal{C})$ there is an l-group $G \in \mathrm{Res}(\mathcal{C})$ and an l-homomorphism ϕ from G onto H. Suppose we are given a map $\alpha: X \longrightarrow H$; select $g_x \in G$ for each $x \in X$ so that $g_x\phi = x\alpha$ and call the assignment $x \longrightarrow g_x\ \beta$; then $\beta\phi = \alpha$. By the previous paragraph β has an extension to an l-homomorphism $\beta^*: F_{\mathcal{C}}(X) \longrightarrow G$. If we let $\alpha^* = \beta^*\phi$ then $x\alpha^* = x\beta^*\phi = x\beta\phi = x\alpha$, for all $x \in X$. Since X generates $F_{\mathcal{C}}(X)$ as an l-subgroup, α^* is uniquely determined by α. Hence $F_{\mathcal{C}}(X)$ is free over X in $\mathrm{Var}(\mathcal{C})$, and the proof is complete. □

Remark. This theorem sheds some light on Weinberg's theorem stating that the free abelian l-groups are subdirect products of integers (see [9]). We cannot derive Weinberg's result as a corollary, for it is precisely that result which guarantees that $\mathrm{Var}(\mathcal{Z})$ is the variety of all abelian l-groups. But our theorem does point out that the theorem of Weinberg is not an oddity.

It becomes interesting then to determine just what variety of l-groups is generated by the class of c-archimedean o-groups. One wonders, in fact, whether this might not be the variety of representable l-groups.

The theorem also suggests that if one is doing work in universal algebra of l-groups one might prefer to work with classes like $\mathrm{Res}(\mathcal{C})$, for a suitable class of o-groups $\mathcal{C}$. According to 3.5, $\mathrm{Res}(\mathcal{C})$ has free l-groups, and a similar argument shows it has free products: let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a family of residually-$\mathcal{C}$ l-groups, and $\overline{G} = \coprod G_\lambda$ be the free product of the G_λ in the variety of representable l-groups. Let $G = \overline{G}/R_{\mathcal{C}}(\overline{G})$ and $\eta\colon \overline{G} \longrightarrow G$ be the canonical map; if $u_\lambda\colon G_\lambda \longrightarrow \overline{G}$ is the λth coprojection, let $v_\lambda = u_\lambda \eta$. For any family $\{\phi_\lambda\colon G_\lambda \longrightarrow H\}$ of l-homomorphisms into a residually-$\mathcal{C}$ l-group H, let $\overline{\phi}\colon \overline{G} \longrightarrow H$ be the unique extension to $\overline{G}$. As in the proof of 3.5, $R_{\mathcal{C}}(\overline{G}) \subseteq \mathrm{Ker}(\overline{\phi})$, and so $\overline{\phi}$ factors through G with an l-homomorphism $\phi\colon G \longrightarrow H$ such that $\eta\phi = \overline{\phi}$. For each $\lambda \in \Lambda$, $v_\lambda \phi = u_\lambda \eta \phi = u_\lambda \overline{\phi} = \phi_\lambda$. This proves that G is the free product of the G_λ in $\mathrm{Res}(\mathcal{C})$ with v_λ as λth coprojection.

We summarize

3.6 **Proposition.** *For any class* $\mathcal{C}$ *the category* $\mathrm{Res}(\mathcal{C})$ *has free products.*

Caution. The free product of residually-$\mathcal{C}$ l-groups in $\mathrm{Res}(\mathcal{C})$ is in general not isomorphic to their free product in $\mathrm{Var}(\mathcal{C})$. It is shown in [7] that the free product of $\mathbf{R}$ with itself in the variety of abelian l-groups is not a subdirect product of reals (it is not even an archimedean l-group!).

One word about Corollary 3.4.1 is in order here: if $\mathcal{C}$ is an arbitrary class the $\mathcal{C}$-radical of a (representable) l-group G might not be the word l-ideal determined by $\mathrm{Var}(\mathcal{C})$, but it is fully invariant nevertheless. This is easily seen by proving that if H is an l-subgroup of G then $R_{\mathcal{C}}(H) \subseteq R_{\mathcal{C}}(G)$; Lemma 3.3 takes care of the remainder of the argument.

All of this seems to suggest that for a given class $\mathcal{C}$ of o-groups there is an elementwise definition of the l-groups in $\mathrm{Res}(\mathcal{C})$ and perhaps even $\mathrm{Para}(\mathcal{C})$. For any representable l-group G, the elements of $R_{\mathcal{C}}(G)$ are the obstruction to G being residually-$\mathcal{C}$; G is residually-$\mathcal{C}$ if and only if it has no such elements. These elements are defined in terms of l-ideal values in G, so one might hope to do better. As we have already indicated (see §2) for the class $\mathcal{Z}$ and the c-archimedean l-groups, the situation in $\mathrm{Para}(\mathcal{C})$ is much more complicated.

Finally, one ought to try to abstract the classes of the type of $\mathrm{Res}(\mathcal{C})$, with an eye on a theorem which specializes (when $\mathcal{C}$ is a full class) to Birkhoff's celebrated theorem on varieties. Any class $\mathcal{X} = \mathrm{Res}(\mathcal{C})$ has the following properties:

(1) $\mathfrak{X}$ is l-subgroup closed, (2) $\mathfrak{X}$ is residually closed, (3) $\mathfrak{X}$ is closed under quotients by polar subgroups, and (4) if $G \in \mathfrak{X}$ is finite valued then every l-homomorphic image of G is in $\mathfrak{X}$.

Conversely, suppose $\mathfrak{X}$ is a class of representable l-groups satisfying (1) through (4) above, and let $\mathcal{C}$ be the class of o-groups in X. It is easy to prove the following result.

3.7 **Proposition.** (i) *$\mathcal{C}$ is closed under taking subgroups and quotients.*

(ii) *$\mathcal{C}$ is residually closed (with respect to o-groups).*

(iii) $\mathrm{Res}(\mathcal{C}) \subseteq \mathfrak{X}$.

(iv) $G \in \mathrm{Hyp}(\mathcal{C})$ *if and only if* $G \in \mathfrak{X}$ *and every l-homomorphic image of* G *is also in* $\mathfrak{X}$.

One wants equality in 3.7(iii); it is readily seen that it does not happen in general: just let $\mathfrak{X} = \mathrm{Para}(\mathcal{C})$ for any residually closed class of o-groups. Evidently then the list of properties for $\mathfrak{X}$ is incomplete. If one adds the requirement: (5) $\mathfrak{X}$ is locally closed (with respect to convex l-subgroups), then $\mathfrak{X} \supseteq \mathrm{Para}(\mathcal{C})$. One wonders whether equality must hold there; that is, do the closure properties labelled (1) through (5) characterize the $\mathrm{Para}(\mathcal{C})$ classes? We know of no counterexamples to the conjecture that this be so.

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